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III.

On the Square Bar Micrometer.

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1. THE use of a square for the micrometric observation of comets was proposed in 1800, by D. Burckhardt, in Zach's *Monatliche Correspondenz*, Vol. I. p. 120. The idea had been suggested to him by a figure of such an instrument in Adam's "Description of Mathematical Instruments." Burckhardt shows the simple manner in which, by means of transits over the sides and the diagonal of the square, the difference of position of two objects can be determined, in any position of the square with reference to the diurnal motion, and its advantage over Bradley's rhomboidal micrometer in this respect. Nearly half a century later, the square was independently proposed by Mr. A. Graham, at the Markree observatory, and a very large use of it was made in the determination of approximate places of stars, for charting purposes. Graham constructs his micrometer by four flat bars of thin steel, laid over each other in the form of a square, and fastened by their projecting ends to a circular diaphragm. An improvement in the construction is to have a similar figure to that formed by the overlaid bars and the diaphragm cut out of a single plate of thin metal, so that all four sides lie in the same focal plane. Such a micrometer has been in use for about twenty years at the Harvard College observatory.

It seems singular that the square bar micrometer has not come into more general use. It appears to me to be preferable to the ring, having over the latter two important advantages, without compensating defects. First, all the transits may be observed over the following edges of the bars, so that the entries and exits are both determined by the phenomenon of *disappearance;* consequently the lengths of the observed chords are largely free from that highly prejudicial source of error, which is so troublesome in the ring micrometer, arising from the difference in estimating transits of such dissimilar objects as stars and comets, at disappearance and reappearance. An additional injurious effect of this cause proceeds, in the case of the ring, from the varying angle which the paths make with its edge.

A second advantage of the square is, that differences of right ascension and declination are observed with equal accuracy, irrespective of the difference of declination of the comet and the star, and of the parts of the micrometer which they traverse. In the ring micrometer, on the contrary, a considerable portion of its circumference cannot be used for declination determinations, and the difficulties which often arise from the inconvenient position of the comet relative to its comparison star, are such as require considerable skill and judgment to overcome, and render it a dangerous tool in the hands of an inexperienced observer.

In a third particular, of the least importance, the square recommends itself by the greater simplicity of the reductions. This advantage obtains fully, however, only when the equatorial is provided with a position-circle, and is in good adjustment; so that the same zero of position, once determined, will serve for observations in all parts of the sky. When the zero has to be determined at each observation, the labor involved fairly offsets this advantage.

From what experience I have had with both instruments I believe that the square bar micrometer may be advantageously substituted for the ring micrometer. As a means to bring it into wider employment, I have thought it would be useful to describe the processes of observation, and to develop the various formulæ of reduction, and this I proceed to do.

2. *Orientation of the square.* — The square is to be adjusted so that one diagonal is parallel either to the real, or to the apparent, diurnal motion. Which of these modes of orientation is adopted by the observer will depend upon his circumstances and instrument. The processes involved in this adjustment, and in the reduction of the observations, differ in the two cases in some important particulars. Their description is deferred for convenience to articles 9 and 10.

3. *Mode of observation, and notation of transits.* — The instants of transit to be noted are those of disappearance of the star or comet behind the edge of each bar. These transits will ordinarily be over those portions of the bars forming the sides of the square; but when the difference of declination is greater than the diagonal, it will be necessary to observe one or both objects over the prolongations of the bars beyond the square, north or south.

In order to deduce general formulæ, applicable without special modification to all observations, inside or outside the square, the following notation is adopted.

Call t_1 the time of the star's transit over the north following or the south following bar (i. e., the bars which intersect at the position angle 90°), whether the transit occurs inside or outside the square; and t_2 its time of transit over the north preceding or the south preceding bars (i. e., those intersecting at 270°), also irrespective of the place on the bar, inside or outside. Take for the comet the same letters accented, so that t'_1 and t'_2 are, respectively, the times of the comet's transit over a following and a preceding bar, as just defined, whether inside or outside. Thus t_2 is numerically greater than t_1, and t'_2 greater than t'_1, when the transit occurs inside, and less when it occurs outside. This notation preserves, algebraically, the relations which represent the geometric conditions for all the positions which the objects can occupy in the field, with reference to each other and to the square; and, if strictly adhered to, render all the formulæ hereafter given entirely general.

4. *Determination of the uncorrected apparent differences of right ascension and declination of two objects.*—By the term *uncorrected differences* are meant those found by assuming the micrometer to be perfectly adjusted, and the paths of the objects to be rectilinear, and also by ignoring the effects of refraction, and of the motion of the comet relative to the star during the observation.

Let α, α' be the apparent right ascensions, and δ, δ' the apparent declinations, of the star and comet, respectively; and let g be the value, in arc, of the diagonal of the square.

The uncorrected difference of right ascension will then be

$$\alpha' - \alpha = \tfrac{1}{2}(t'_1 + t'_2) - \tfrac{1}{2}(t_1 + t_2). \tag{1}$$

Calling d the difference of declination of the star and the centre of the square, we have

$$d = \pm \frac{g}{2} \mp \frac{t_2 - t_1}{2}\, 15 \cos \delta\,;$$

and, similarly for the comet, (2)

$$d' = \pm \frac{g'}{2} \mp \frac{t'_2 - t'_1}{2}\, 15 \cos \delta'\,;$$

where g and g' are of course numerically equal, but the upper or under sign is to be used, according as the path is north or south of the centre of the square. The difference of these equations gives

$$\delta' - \delta = d' - d = \pm \frac{g'}{2} \mp \frac{g}{2} - \frac{15}{2}\left[\pm (t'_2 - t'_1) \cos \delta' \mp (t_2 - t_1) \cos \delta\right], \tag{3}$$

which is the uncorrected difference of declination of the comet and the star. This general expression assumes the following forms for the special cases indicated below.

$$\left.\begin{array}{l}\text{Comet and star both north}\\ \text{Comet and star both south}\\ \text{Comet north, star south}\\ \text{Comet south, star north}\end{array}\right\}\delta'-\delta=\left\{\begin{array}{r}-\tfrac{15}{2}\left[(t'_2-t'_1)\cos\delta'-(t_2-t_1)\cos\delta\right]\\ +\tfrac{15}{2}\left[(t'_2-t'_1)\cos\delta'-(t_2-t_1)\cos\delta\right]\\ +g-\tfrac{15}{2}\left[(t'_2-t'_1)\cos\delta'+(t_2-t_1)\cos\delta\right]\\ -g+\tfrac{15}{2}\left[(t'_2-t'_1)\cos\delta'+(t_2-t_1)\cos\delta\right]\end{array}\right. \tag{4}$$

For moderate declinations, say under 40°, we may take $\delta_0 = \frac{1}{2}(\delta' + \delta)$ and write equation (3) with sufficient practical accuracy

$$\delta' - \delta = d' - d = \pm\frac{g'}{2} \mp \frac{g}{2} - \frac{15}{2}\cos\delta_0\left[\pm(t'_2 - t'_1) \mp (t_2 - t_1)\right]; \tag{5}$$

and the special forms (4) in a correspondingly simplified manner.

5. *Correction for curvature.*—When the declination of the objects observed is considerable, the difference of declination found by (3) requires a correction for the curvature of their paths, which may be obtained in the following manner:—

The value of d given by (2) corresponds to the distance from the centre of the square of the point of intersection of a great circle joining the places of the star's entry and exit, with the diagonal joining the north and south angles of the square. The actual path of the star, being a small circle, cuts the diagonal at a small distance, which we will call x, from the great circle. From the triangle formed by the pole, the star when on the bar, and the intersection of the great circle and diagonal, we have

$$\cot(\delta + x) = \cos\tfrac{15}{2}(t_2 - t_1)\cot\delta.$$

Developing this in the known way we find

$$x = \frac{\tan^2\frac{15}{4}(t_2 - t_1)}{\sin 1''}\sin 2\,\delta + \frac{1}{2}\,\frac{\tan^4\frac{15}{4}(t_2 - t_1)}{\sin 1''}\sin 4\,\delta + \ldots.$$

which is subtractive from the uncorrected value of d in (2). For the comet we have a precisely similar equation with accents. Hence, neglecting the 4th and higher powers, and employing the mean declination δ_0, we have, nearly, putting sines for tangents,

$$\delta' - \delta = (d' - d) - \frac{2\sin^2\frac{15}{4}(t'_2 - t'_1) - 2\sin^2\frac{15}{4}(t_2 - t_1)}{\sin 1''}\cos\delta_0\sin\delta_0,$$

where the last term is the required correction for curvature. If we take values m' and m from Table V., Chauvenet's Astronomy, Vol. II., using the side arguments $\frac{1}{2}(t'_2 - t'_1)$ and $\frac{1}{2}(t_2 - t_1)$, we can write this equation very simply,

$$\delta' - \delta = (d' - d) - (m' - m) \cos \delta_0 \sin \delta_0 ; \tag{6}$$

where $(d' - d)$ is the uncorrected value given by (3). For a high declination we can, without sensible error, write $\cos \delta_0$ for $\cos \delta_0 \sin \delta_0$.

For very high declinations, say above 85°, we should, for greater accuracy in computing $(d' - d)$, substitute the sines for the arcs described by the star and comet; so that instead of (6) we have the more rigorous expression,

$$\delta' - \delta = \pm \frac{g'}{2} \mp \frac{g}{2} - \frac{\pm \sin \frac{15}{2}(t'_2 - t'_1) \cos \delta' \mp \sin \frac{15}{2}(t_2 - t_1) \cos \delta}{\sin 1''} - (m' - m) \cos \delta_0. \tag{7}$$

It will, however, seldom be necessary to employ this equation, since in practice, for stars so near the pole as to require it, the use of the square or of any micrometer dependent on transits, would be abandoned as inconvenient.

6. *Correction for proper motion.* — Let $\Delta \alpha'$ be the increase in time-seconds of the comet's right ascension, and $\Delta \delta'$ the increase in arc-seconds of its declination (positive for northerly motion), each during one sidereal second. It is manifest that the time of the comet's arrival at both preceding and following bars is accelerated or retarded by the amount necessary for it to describe an arc equal to the comet's motion in declination during half the duration of the transit. Consequently, to the uncorrected difference of right ascension given by (1) is to be added the correction

$$\pm \frac{t'_2 - t'_1}{2} \frac{\Delta \delta'}{15 \cos \delta'} ; \tag{8}$$

where the upper or under sign is to be used, according as the comet passes north or south of the centre.

Another effect of the proper motion is to alter the length of the chord described in the interval $(t'_2 - t'_1)$ by an amount equal to the comet's motion in right ascension during that interval. The corrected length of the chord is $(t'_2 - t'_1)(1 - \Delta \alpha') \cos \delta'$, whence the uncorrected difference of declination found by (3) requires the correction

$$\pm \frac{t'_2 - t'_1}{2} \Delta \alpha' . 15 \cos \delta' ; \tag{9}$$

the double sign having the same significance as before.

7. *Correction for refraction when the square is adjusted to the true diurnal motion.*— Let it be assumed that the square is oriented with reference to the true diurnal motion. Let θ be the sidereal time when the star is actually on the north and south diagonal, and $\Delta_1 \alpha$, $\Delta_2 \alpha$, the refraction in right ascension, and $\Delta_1 \delta$, $\Delta_2 \delta$, the refraction in declination, at entry and exit, respectively. Let the angular distance, expressed in time, of the middle points of the true and apparent paths from the diagonal be denoted by I and H, respectively. Then we have

$$\frac{t_2 + t_1}{2} - \theta = I \sec \delta = \frac{\Delta_2 \alpha + \Delta_1 \alpha}{2} - H \sec \delta.$$

It will be seen by consideration of the geometrical conditions that

$$H = \pm \frac{1}{15} \cdot \frac{\Delta_2 \delta - \Delta_1 \delta}{2},$$

the upper or under sign corresponding to positions north or south of the centre. Hence, since a similar equation obtains for the comet, with accents,

$$\frac{t_2 + t_1}{2} - \theta = \frac{\Delta_2 \alpha + \Delta_1 \alpha}{2} \mp \frac{1}{15} \cdot \frac{\Delta_2 \delta - \Delta_1 \delta}{2} \sec \delta,$$

$$\frac{t'_2 + t'_1}{2} - \theta' = \frac{\Delta_2 \alpha' + \Delta_1 \alpha'}{2} \mp \frac{1}{15} \cdot \frac{\Delta_2 \delta' - \Delta_1 \delta'}{2} \sec \delta'.$$

The difference of the first members gives

$$(\theta' - \theta) - [\tfrac{1}{2}(t'_2 + t'_1) - \tfrac{1}{2}(t_2 + t_1)].$$

The first term of this being the true difference of right ascension, and the second term the apparent difference (see equation (1)), we have manifestly from the difference of the second members of the above equations, the correction for differential refraction in right ascension,

$$\Delta(\alpha' - \alpha) = \tfrac{1}{2}[(\Delta_2 \alpha + \Delta_1 \alpha) - (\Delta_2 \alpha' + \Delta_1 \alpha')] + \tfrac{1}{2}[\pm(\Delta_2 \delta' - \Delta_1 \delta') \mp (\Delta_2 \delta - \Delta_1 \delta)] \frac{1}{15 \cos \delta_0}. \quad (10)$$

To get the effect of refraction on the difference of declinations, we observe that the length of the apparent path described by the star is

$$15 \cos(\delta) [(t_2 - t_1) - (\Delta_2 \alpha - \Delta_1 \alpha)],$$

where (δ) is the apparent[1] declination; and that, although not perpendicular to the diagonal, it will be equal in length, nearly, to a line drawn perpendicular to the diagonal at its intersection with the apparent path. Hence the distance of the middle point of the apparent path from the centre of the square will be

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[1] In this formula as originally written I had inserted the *true* declination. The manuscript having been transmitted to Professor Young, and carefully examined by his assistant, Mr. Malcolm McNeill, they kindly pointed out that, rigorously, the apparent declination should here be used. The effect of the difference is, of course, very small; in fact, inappreciable except for high declinations, near the horizon.

$$\pm \frac{g}{2} \mp \frac{15}{2} \cos(\delta) \left[(t_2 - t_1) - (\Delta_2 a - \Delta_1 a)\right];$$

and, since the distance between the apparent and true paths is $\frac{1}{2}(\Delta_2 \delta + \Delta_1 \delta)$, we have, putting $D =$ the declination of the centre,

$$\delta - D = \pm \frac{g}{2} \mp \frac{15}{2} \cos(\delta) \left[(t_2 - t_1) - (\Delta_2 a - \Delta_1 a)\right] - \tfrac{1}{2}(\Delta_2 \delta + \Delta_1 \delta).$$

Putting $(\delta) = \delta + \frac{1}{2}(\Delta_2 \delta + \Delta_1 \delta)$ and developing, neglecting squares of the refractions,

$$\delta - D = \pm \frac{g'}{2} \mp \frac{15}{2} \cos \delta \left[(t_2 - t_1) - (\Delta_2 a - \Delta_1 a)\right] - \tfrac{1}{2}(\Delta_2 \delta + \Delta_1 \delta)\left[1 \mp \tfrac{15}{2} \sin 1'' \sin \delta \, (t_2 - t_1)\right].$$

For the comet a similar equation obtains, with accents. Taking the difference and remembering the relation (3) we get $(\delta' - \delta) - (d' - d)$, or the correction for refraction in declination,

$$\begin{aligned}\Delta(\delta' - \delta) = \tfrac{15}{2} \cos \delta_0 \left[\pm (\Delta_2 a' - \Delta_1 a') \mp (\Delta_2 a - \Delta_1 a)\right] - \tfrac{1}{2}\left[(\Delta_2 \delta' + \Delta_1 \delta') - (\Delta_2 \delta + \Delta_1 \delta)\right] \\ + \tfrac{15}{2} \sin \delta \sin 1'' \left[\pm \frac{t'_2 - t'_1}{2}(\Delta_2 \delta' + \Delta_1 \delta') \mp \frac{t_2 - t_1}{2}(\Delta_2 \delta + \Delta_1 \delta)\right]. \qquad (11)\end{aligned}$$

It remains to reduce (10) and (11) to a form convenient for direct computation.

Assuming the refraction to vary at a uniform rate over the field of view, the first term of (10) is manifestly the difference of the refractions in right ascension for the star and the comet, at the middle points of their chords; and the second term is one half the difference between the changes in the refraction in declination for the comet and star, during the interval between their respective entries and exits, reduced to time at the given declination. Hence, if ζ and ζ' be the true zenith distances of the star and comet, and q and q' be their parallactic angles, we have (Chauvenet's Astronomy, Vol. II., equation 234),

$$\begin{aligned}&\tfrac{1}{2}(\Delta_2 a + \Delta_1 a) - \tfrac{1}{2}(\Delta_2 a' + \Delta_1 a') = k \frac{\tan \zeta \, \sin q}{15 \cos \delta} - k' \frac{\tan \zeta' \, \sin q'}{15 \cos \delta'}, \\ &\frac{1}{15 \cos \delta_0} \cdot \tfrac{1}{2}\left[\pm (\Delta_2 \delta' - \Delta_1 \delta') \mp (\Delta_2 \delta - \Delta_1 \delta)\right] = \qquad (12) \\ &\qquad \frac{1}{2 \cos \delta_0} \cdot \left[\mp (t'_2 - t'_1) \, k' \tan \zeta' \cos q' \pm (t_2 - t_1) \, k \tan \zeta \cos q\right].\end{aligned}$$

Take a common value for ζ and q corresponding to the centre of the square, and Bessel's coefficient of differential refraction κ for this point. Put also, for brevity,

$$\mu = \frac{\tan \zeta \sin q}{\cos \delta}, \quad \nu = \tan \zeta \cos q. \tag{13}$$

Then the addition of (12) gives the value of (10), which may be written (on the general principle that, when u is a function of x, the change of u, corresponding to a small finite change $x' - x$, is $\frac{du}{dx}(x' - x)$),

$$\Delta(\alpha' - \alpha) = -\tfrac{1}{15}\kappa(\delta' - \delta)\frac{d\mu}{d\delta} - \kappa \sec \delta_0 \left[\pm \frac{t'_2 - t'_1}{2} \mp \frac{t_2 - t_1}{2}\right]\frac{d\nu}{d\alpha}. \tag{14}$$

The differential coefficients in (14) may be got from the known fundamental relations,

$$\begin{aligned} \cos \zeta &= \sin \phi \sin \delta + \cos \phi \cos \delta \cos t; \\ \sin \zeta \cos q &= \sin \phi \cos \delta - \cos \phi \sin \delta \cos t; \\ \sin \zeta \sin q &= \cos \phi \sin t. \end{aligned} \tag{15}$$

Differentiating these with respect to δ,

$$\begin{aligned} d(\cos \zeta) &= -\sin \zeta \, d\zeta = \sin \zeta \cos q \, d\delta; \\ d(\sin \zeta \cos q) &= -\cos \zeta \, d\delta; \\ d(\sin \zeta \sin q) &= 0; \end{aligned}$$

and substituting in the value of $\frac{d\mu}{d\delta}$ obtained from (13) we get

$$\frac{d\mu}{d\delta} = -\left[\tfrac{1}{2}\tan^2 \zeta \sin 2q - \tan \zeta \sin q \tan \delta\right] \sec \delta. \tag{16}$$

Differentiating the first and second of (15) with reference to t,

$$\begin{aligned} d(\cos \zeta) &= -\sin \zeta \, d\zeta = -\sin \zeta \sin q \cos \delta \, dt; \\ d(\sin \zeta \cos q) &= \sin \zeta \sin q \sin \delta \, dt; \end{aligned}$$

and substituting in $\frac{d\nu}{dt}$ from (13), we get

$$\frac{d\nu}{dt} = \tfrac{1}{2}\tan^2 \zeta \sin 2q \cos \delta + \tan \zeta \sin q \sin \delta. \tag{17}$$

Substituting (16) and (17) in (14), remembering that $d\alpha = -dt$ and also the relation (5), we derive, finally, the refraction correction in right ascension,

$$\Delta(\alpha' - \alpha) = \frac{\kappa}{15 \cos \delta_0} \left(\pm \frac{g'}{2} \mp \frac{g}{2}\right) \left(\tfrac{1}{2} \tan^2 \zeta \sin 2q + \tan \zeta \sin q \tan \delta\right)$$

$$- \frac{2\kappa}{15 \cos \delta_0} (d' - d) \tan \zeta \sin q \tan \delta. \qquad (18)$$

To reduce the declination correction we note that the first term of (11) is equivalent to one half the difference between the changes in refraction in right ascension for the comet and star in the interval between entry and exit, reduced to arc of a great circle; while the second term is the difference of the refractions in declination, at the middle points of their chords. So that, in a similar way to that before used for (10), we transform the first two terms of (11) into

$$15 \cos \delta \, \kappa \left(\mp \frac{t'_2 - t'_1}{2} \pm \frac{t_2 - t_1}{2}\right) \frac{d\mu}{d\alpha} - \kappa (\delta' - \delta) \frac{d\nu}{d\delta}.$$

With regard to the third term of (11), an examination shows that its numerical value must be very small, and that it may be expressed very nearly by

$$15 \sin \delta \sin 1'' \left[\pm \frac{t'_2 - t'_1}{2} \mp \frac{t_2 - t_1}{2}\right] \Delta \delta,$$

where $\Delta \delta$ is equal to the refraction in declination for a star at the centre of the field, and is expressed by $k \tan \zeta \cos q$; for which we may substitute without sensible error, $\kappa \operatorname{cosec} 1'' \tan \zeta \cos q$, as is manifest from the way κ is derived from k. Therefore the third term of (11) may be written

$$15 \cos \delta \, \kappa \left[\pm \frac{t'_2 - t'_1}{2} \mp \frac{t_2 - t_1}{2}\right] \nu \tan \delta.$$

Adding this to the above value of the first two terms of (11), and remembering that, from (5),

$$15 \cos \delta \left[\pm \frac{t'_2 - t'_1}{2} \mp \frac{t_2 - t_1}{2}\right] = \pm \frac{g'}{2} \mp \frac{g}{2} - (\delta' - \delta),$$

we obtain

$$\Delta(\delta' - \delta) = \left[\pm \frac{g'}{2} \mp \frac{g}{2} - (\delta' - \delta)\right] \kappa \left(\nu \tan \delta - \frac{d\mu}{d\alpha}\right) - \kappa (\delta' - \delta) \frac{d\nu}{d\delta}. \qquad (19)$$

Differentiate the first and third of (15) with respect to t, and the second with respect to δ,

$$d(\cos \zeta) = - \sin \zeta \, d\zeta = - \cos \phi \cos \delta \sin t \, dt;$$

$$d(\sin \zeta \sin q) = \cos \phi \cos t \, dt;$$

$$d(\sin \zeta \cos q) = - \cos \zeta \, d\delta.$$

Also multiply the first of (15) by $\cos\delta$ and the second by $\sin\delta$, subtract and reduce, and we get

$$\frac{\cos\phi\cos t}{\cos\delta\ \cos\zeta} = 1 - \tan\zeta\cos q\tan\delta.$$

Employing these relations in the values of $\frac{d\mu}{dt}$ and $\frac{d\nu}{d\delta}$ from (13) we get,

$$\frac{d\mu}{d\alpha} = -\tan^2\zeta\sin^2 q + \tan\zeta\cos q\tan\delta - 1;$$

$$\frac{d\nu}{d\delta} = -1 - \tan^2\zeta\cos^2 q;$$

which, introduced into (19) give, finally, for the refraction correction in declination,

$$\Delta(\delta' - \delta) = \kappa\left(\pm\frac{g'}{2} \mp \frac{g}{2}\right)(\tan^2\zeta\sin^2 q + 1) + \kappa(d' - d)\tan^2\zeta\cos 2q. \tag{20}$$

Equations (18) and (20) are, therefore, the expressions (general for any positions of the comet and star, inside or outside the square) by which the refraction corrections may be accurately computed, when the square has been oriented with respect to the *true* diurnal motion. We can often substitute for these the simpler approximate expressions, which involve the terms in $\tan^2\zeta$ only,

$$\left.\begin{aligned}\Delta(\alpha' - \alpha) &= \frac{\kappa}{30\cos\delta_0}\left(\pm\frac{g'}{2} \mp \frac{g}{2}\right)\tan^2\zeta\sin 2q;\\ \Delta(\delta' - \delta) &= \kappa\left(\pm\frac{g'}{2} \mp \frac{g}{2}\right)\tan^2\zeta\sin^2 q + \kappa(d' - d)\tan^2\zeta\cos 2q;\end{aligned}\right\} \tag{21}$$

where, it will be noticed, the whole correction in right ascension, and the first term of that in declination, disappear when the star and the comet are observed in the same half of the square.

The precaution should always be exercised, before using the approximate values given by (21), to see, by a general inspection of the case in hand, whether the smaller terms of which the more rigorous equations (18) and (20) take account, are fairly negligible.

The constant κ can be computed from section *C* of Table II., in Vol. II. of Chauvenet's Astronomy, if necessary for the particular state of the air, by article 295, which also gives the formulæ for ζ and q. Except for extreme atmospheric conditions, or near the horizon, however, it will be sufficiently accurate to take $\log\kappa$ for the mean state of the air, or equal to $\log\alpha''$ of the table.

8. *Correction for refraction when the square is adjusted to the apparent diurnal motion.*—In this case the above formulæ for the right ascension, (18), or the first of (21), will require modification, on account of the effect of the difference in position angle between the true and the apparent motion. Call Δp the angle which the apparent path of a star, traversing the square, makes with its true path. This angle will be the ratio which the change in refraction in declination bears to the length of the chord, or

$$\Delta p = \frac{1}{15 \sin 1''} \cdot \frac{\Delta_2 \delta - \Delta_1 \delta}{\cos \delta \left[(t_2 - t_1) - (\Delta_2 \alpha - \Delta_1 \alpha) \right]}.$$

By the preceding article we find

$$\Delta_2 \delta - \Delta_1 \delta = 15 \kappa (t_2 - t_1) \frac{d v}{d t};$$

consequently we have, neglecting terms of the second order,

$$\Delta p \doteq \frac{\kappa}{\sin 1''} \left[\tfrac{1}{2} \tan^2 \zeta \sin 2 q + \tan \zeta \sin q \tan \delta \right]. \tag{22}$$

Substituting this value for p in (25) we obtain

$$\frac{\kappa}{15 \cos \delta} \left[2 (d' - d) - \left(\pm \frac{g'}{2} \mp \frac{g}{2} \right) \right] \left(\tfrac{1}{2} \tan^2 \zeta \sin 2 q + \tan \zeta \sin q \tan \delta \right), \tag{22*}$$

as the correction to the difference of right ascension obtained by assuming the square to have been adjusted to the true diurnal motion. Adding (18) and (22*) we have

$$\Delta (\alpha' - \alpha) = \frac{\kappa}{15 \cos \delta_0} (d' - d) \tan^2 \zeta \sin 2 q, \tag{23}$$

which is the correction for refraction in right ascension when the square is oriented to the apparent diurnal motion, by the methods of article 10.

The declination correction is unaffected by the mode of adjustment.

It should be borne in mind that (23) has been deduced by assuming that the values of κ, ζ, and q are identical in (18) and (22); in other words, that the adjustment has been made with the telescope pointed in the same direction as for the comparisons of the comet and star. Ordinarily this assumption is nearly enough true; but near the horizon, for considerable hour angles, it may sometimes be requisite for accuracy to correct for refraction in right ascension by applying (18) and (22) separately, with the particular values of ζ and q pertaining to the comet comparisons and to the adjustment observations, respectively. With care, however, the necessity for this may almost always be avoided.

9. *Orientation of the square to the real diurnal motion.* — Suppose the telescope to be equatorially mounted, provided with a micrometer position circle, and in good adjustment, so that the instrumental pole is very close to the true pole, say within a minute of arc, as it should be in a well mounted instrument. Turn to a star on or very near the meridian, and set the square by the position circle so that the star traverses the diagonal, or parallel to it, as nearly as possible. Call the reading of the position circle P, and observe the times of transit as described in article 12. Since on the meridian Δp disappears, $p = p'$, and is found by equation (32). The reading $P - p$ will be the true zero of position at which the position circle should be thereafter set to orient the square to the true diurnal motion. If the equatorial is in good adjustment it will answer for all parts of the sky. If not, the correction due to the equatorial errors may be determined as shown in the text books (Chauvenet, II., p. 261).

If it be inconvenient to make this adjustment on the meridian, an extra-meridian determination can be made, the correction for refraction, Δp, being found by (22) and p' by (32). Then the reading for the setting to true diurnal motion will be $P - (p' + \Delta p)$.

It is scarcely necessary to say that the adapting tube and micrometer should be so marked, or fitted to each other, that the latter may always be inserted at a determinate position angle in the telescope.

If the telescope be not provided with a position circle the adjustment can still be made nearly as above described, with a little additional trouble, by several approximations, the tubes being carefully marked when the true position zero is found.

The refraction corrections to be used in reducing observations made with the square adjusted by this article are (18) and (20), or (21).

10. *Orientation of the square to the apparent diurnal motion.* — The adjustment by the previous article is to be preferred, on the score of convenience, when the micrometer is to be actively employed, as it need be made once only for all observations. Where, however, the telescope is not mounted equatorially, or the equatorial errors are unknown, or for other reasons, it may be necessary to adjust the square on the occasion of each observation. The orientation will then be more conveniently made with respect to the apparent diurnal motion. Several cases may arise.

Case I. Equatorial provided with position micrometer circle, instrumental adjustments unknown. Point the telescope as nearly as possible to the hour angle and zenith distance at which the comparisons of the comet and star are to be made immediately subsequent, and set the position circle so that a star will skirt along one side of the square, parallel to it. With several trials the position of parallelism can be approximated to quite closely. Then turn the position circle 45° and proceed with the comet comparisons. This, while not a refined method of observation, will answer well enough when the comet and its comparison star are nearly on a parallel; as the effect of an error in setting is then small.

Case II. For greater accuracy than the preceding method affords, when the difference of declination of the objects is considerable, or in the case there is no position circle, we can determine the error of setting as follows. Point the telescope as before, as nearly as possible in the direction in which the comet comparisons are to be made. Set the square by trial approximately so that a star will traverse the diagonal, or parallel to it. Observe the transit of a star as described in article 12, and deduce p' by equation (32). Then proceed with the comet observations, and in their reduction apply the correction computed by (25) or (26), putting p' for p.

Case III. A better way of proceeding is to arrange the observations so that p' can be determined simultaneously with $\alpha' - \alpha$, and $\delta' - \delta$. This method can be adopted, also, when the telescope is not mounted equatorially. If $\delta' - \delta$ is not much greater than the half diagonal, the comparisons of the comet and star may be so arranged that the latter passes near the centre and its transits observed as described in article 12. This will determine p' by (32), whence we get the correction for position zero by (25). If $\delta' - \delta$ is too great to allow this mode of observation, any star which traverses the central portion of the square during the comparisons, may be observed in conjunction with the comet and its comparison star, and p' thus determined.

The refraction corrections to be used in reducing observations made with the square adjusted by the methods of this article, are (23) for the difference of right ascensions, and (20), or the second of (21), for the difference of declinations.

11. *Correction for position zero.* — Let p expressed in arc be the small angle which the diagonal makes with the hour circle passing through the centre of the square, or the position angle of the north angle of the square. It is evident that the time when the star is on the diagonal is

$$\frac{t_2 + t_1}{2} - \frac{p \sin 1''}{15 \cos \delta} \left(\pm \frac{g}{2} - d \right),$$

and that to this must be added $\frac{p \sin 1''}{15 \cos \delta} d$, to get the time of crossing the hour circle passing through the centre of the square. Consequently the true difference of right ascension of a comet and star will be

$$\alpha' - \alpha = \tfrac{1}{2}(t'_2 + t'_1) - \tfrac{1}{2}(t_2 + t_1) + \frac{p \sin 1''}{15 \cos \delta}\left[2(d' - d) \mp \frac{g'}{2} \pm \frac{g}{2}\right]; \tag{24}$$

whence the correction to be added to the uncorrected difference of right ascension obtained by (1) is

$$\frac{p \sin 1''}{15 \cos \delta}\left[2(d' - d) - \left(\pm \frac{g'}{2} \mp \frac{g}{2}\right)\right] \tag{25}$$

or the equally convenient form

$$p \sin 1''\left[\frac{\pm \frac{1}{2} g' \mp \frac{1}{2} g}{15 \cos \delta} \mp (t'_2 - t'_1) \pm (t_2 - t_1)\right]. \tag{26}$$

To get the accurate expression for the difference of declination it is to be noted that the distance from the angle of the square to the intersection of the star's path with the diagonal is $\pm \frac{g}{2} - d \sec p$; also that the triangle formed by the sides of the square and the star's path gives by Chauvenet's Trigonometry, equation (282),

$$\pm \frac{g}{2} - d \sec p = \pm\ (t_2 - t_1)\, 15 \cos \delta \sin(45^\circ + p) \sin(45^\circ - p);$$

whence

$$d = \pm \frac{g}{2} \cos p \mp \frac{t_2 - t_1}{2}\, 15 \cos \delta \frac{\cos 2p}{\cos p}, \tag{27}$$

the difference of which and a similar equation for the comet gives the accurate difference of declinations

$$\delta' - \delta = \left(\pm \frac{g'}{2} \mp \frac{g}{2}\right) \cos p - 15\left[\pm \frac{t'_2 - t'_1}{2} \cos \delta' \mp \frac{t_2 - t_1}{2} \cos \delta\right] \frac{\cos 2p}{\cos p}. \tag{28}$$

This is a perfectly rigorous formula, whatever the value of p, and can be used in any position of the square, should occasion arise, provided neither object passes from the northern to the southern half of the square, or *vice versa*, during its transit. In general, however, p is small, and we can develop $\cos p$ and $\frac{\cos 2p}{\cos p}$ in series, neglecting 3^d^ and higher powers. Thus we obtain the correction to be added to the result found by (3),

$$p^2 \sin^2 1''\left[\frac{3}{2}\left(\pm \frac{t'_2 - t'_1}{2} \mp \frac{t_2 - t_1}{2}\right) 15 \cos \delta - \frac{1}{2}\left(\pm \frac{g'}{2} \mp \frac{g}{2}\right)\right], \tag{29}$$

or its equivalent,

$$p^2 \sin^2 1''\left[\left(\pm \frac{t'_2 - t'_1}{2} \mp \frac{t_2 - t_1}{2}\right) 15 \cos \delta - \tfrac{1}{2}(d' - d)\right]. \tag{30}$$

This correction is in practice exceedingly small. Thus, for a square with 20 minutes of arc diagonal, and $p = 1^\circ$, the maximum value amounts to less than $0''.2$. The correction in declination on account of position zero may therefore, in general, be neglected.

12. *Determination of position zero.* — The angle p, as defined in article 11, is equivalent to the deviation of the diagonal lying on the parallel from the direction of the *true* diurnal motion; the angle p' is the deviation from the *apparent* motion; and Δp is the difference between the two, due to the refraction; so that

$$p = p' + \Delta p. \tag{31}$$

The value of Δp has been already given, equation (22); that of p' may be found by observation of a star over the central portions of the square, by noting the transits not only over the sides of the square, but also over the prolongations of the bars forming the sides at the preceding and following angles. Let t_0 and t_3 be the times of disappearance at the prolongation of the following and preceding bars, respectively, the notation for the transits over the sides being unchanged, or t_1 and t_2. Thus the order in which the four transits occur is t_0, t_1, t_2, t_3, over the following prolongation, the following side, the preceding side, and the preceding prolongation, respectively. Then it is obvious, without further explanation, that the value of p' will be given by

$$p' = \pm \frac{15 \cos \delta}{g \sin 1''} \left[\frac{t_3 + t_0}{2} - \frac{t_2 + t_1}{2} \right]; \tag{32}$$

the upper or under sign to be used, according as the transit is north or south of the centre.

13. *Determination of the diagonal of the square.* — The transits of a star whose declination is only approximately known, observed as described in the preceding article, will furnish the value of the diagonal. Thus the transits over the sides give, equation (27),

$$g = \pm \frac{2d}{\cos p} + (t_2 - t_1)\, 15 \cos \delta \frac{\cos 2p}{\cos^2 p},$$

and the transits over the prolongations,

$$g = \mp \frac{2d}{\cos p} + (t_3 - t_0)\, 15 \cos \delta \frac{\cos 2p}{\cos^2 p};$$

from the sum of which we derive

$$g = [(t_3 - t_0) + (t_2 - t_1)]\, \tfrac{15}{2} \cos \delta - g p^2 \sin^2 1'', \tag{33}$$

which is the apparent value of the diagonal, as affected by the refraction. The term in p^2 is inappreciable for small values of p. When it cannot be neglected it can be determined by means of (32).

The apparent value of g found from (33) may be freed from refraction by applying the correction

$$\Delta g = - g\,\kappa\,(\tan^2 \zeta \sin^2 q - \tan \zeta \cos q \tan \delta + 1). \tag{34}$$

If the observations for determining the diagonal are made upon equatorial stars, in the meridian, the refraction correction reduces to

$$\Delta g = - g\,\kappa, \tag{35}$$

and can thus be allowed for in a very simple manner. The diagonal may also be found by means of pairs of stars whose differences of declination are accurately known, by means of the relation (3), or, for the highest precision, (28), employing the apparent differences of declination. As this method possesses no advantages over that above presented, and the procedure is obvious, it is not worth while to give details.

14. *Deformation of the square.* — In what has preceded it has been assumed that the micrometer is a perfect square. It should be tested in this respect either by measurement, or by observation of transits of equatorial stars near the meridian. Thus any perceptible inequality of the diagonals can be ascertained by the preceding article, while the equality of the sides may be certified by placing each in succession parallel to the diurnal motion, and observing transits across the adjacent sides near its intersection with them.

If the square has been carefully constructed the errors should be very small. The effect of any tendency to rectangular or rhomboidal form will be eliminated from observed differences of right ascensions and declination, if the comparisons are taken with the square placed first with one, and then with the other diagonal, parallel to the diurnal motion, an equal number of comparisons in each position.

15. *Numerical illustrations.* — In the following examples of the application of the preceding formulæ, some of the corrections are very small and, in practice, would be neglected, but are here computed for illustration.

Example I. — Great comet 1881, III., observed at Harvard College observatory ($\phi = 42^\circ\, 22'.8$), June 29, 1881. Square assumed adjusted to *true* diurnal motion. Diagonal, $889''.8$. Approximate $\delta = 66^\circ\, 10'.5$, $\delta' = 66^\circ\, 1'.5$. $\zeta = 67^\circ 42'$, $q = + 29^\circ.2$. Comet's motion in a second, $+ 0^s.0049$, and $+ 0''.101$.

OBSERVED TRANSITS.

Comparison.	Comet South.		Star North.	
	t'_1.	t'_2.	t_1.	t_2.
	h. m. s.	m. s.	m. s.	m. s.
1	15 45 20.3	46 11.1	42 58.2	43 50.3
2	49 16.9	50 17.2	46 59.1	47 49.7
3	53 17.0	54 26.4	51 2.6	51 52.5
4	57 53.0	59 13.1	55 43.4	56 32.2

REDUCTION; EQUATION (1) AND FOURTH OF EQUATION (4).

Comp.	$\frac{t'_1+t'_2}{2}$.	$\frac{t_1+t_2}{2}$.	$a'-a$.	$t'_2-t'_1$.	t_2-t_1.	
	h. m. s.	m. s.	m. s.	s.	s.	
1	15 45 45.70	43 24.25	+ 2 21.45	50.8	52.1	
2	49 47.05	47 24.40	22.65	60.3	50.6	
3	53 51.70	51 27.55	24.15	69.4	49.9	
4	58 33.05	56 7.80	25.25	80.1	48.8	
Sid. chron.	15 51 59.38 ;		+ 2 23.38	65.15	50.35	
Chron. corr.	− 16 48.7		log (t_2-t_1)	1.81391	1.70200	
Sid. time,	15 35 10.7		log $\frac{15}{2}$	0.87506	0.87506	
Reduction,	6 33 5.0		log cos δ	9.60889	9.60632	
				2.29786	2.18338	
				+ 198.55	+ 152.54	= + 351.09
						− 889.80
Cam. M. T.	$9^{\text{h.}}\ 2^{\text{m.}}\ 5^{\text{s.}}.7$;		$a'-a=+2^{\text{m.}}\ 23^{\text{s}}.38$;	$\delta'-\delta=-8'\ 58''.71$		= − 538.71

REFRACTION CORRECTION.

CONSTANTS.

log $(-g)$	n 2.949	log tan ζ	0.387
log $(d'-d)$	n 2.732	log $\tan^2\zeta$	0.774
log cos δ_0	9.608	log sin q	9.686
log tan δ_0	0.353	log sin $2q$	9.930
log κ	6.435	log cos $2q$	9.719

RIGHT ASCENSION CORRECTION, EQUATION (18).

$\frac{1}{2}\tan^2\zeta\sin 2q$	$+2.53$	0.403
$\tan\zeta\sin q\tan\delta$	$+2.67$	0.426
	$+5.20$	0.716
$-g\kappa$		$n\,9.384$
a. c. $(15\cos\delta)$		9.216
1st term	-0.207	$n\,9.316$
$\tan\zeta\sin q\tan\delta$		0.426
$-2\kappa(d'-d)$		9.468
a. c. $(15\cos\delta)$		9.216
2d term	$+0.129$	9.110
$\Delta(\alpha'-\alpha)$	$-0^{s}.078$	

DECLINATION CORRECTION, EQUATION (20).

$\tan^2\sin^2 q$	$+1.40$	0.146
	$+1.00$	
	$+2.40$	0.380
$-g\kappa$		$n\,9.384$
1st term	-0.58	$n\,9.764$
$\tan^2\zeta\cos 2q$		0.493
$\kappa(d'-d)$		$n\,9.167$
2d term	-0.46	$n\,9.660$
$\Delta(\delta'-\delta)$	$-1''.04$	

CURVATURE CORRECTION, EQUATION (6).

$\frac{1}{2}(t'_2-t'_1) = 32^{s}.6$: table V. gives $m' = 0''.58$

$\frac{1}{2}(t_2-t_1) = 25.2$: table V. gives $m = 0.34$

$\cos\delta\sin\delta = 0.37 \times$ $m'-m = 0.24$

Correction $= -0''.09$.

PROPER MOTION CORRECTION, EQUATIONS (8) AND (9).

$\Delta\alpha'$	$+.0049$	7.690
$\Delta\delta'$	.101	9.004
$-\frac{1}{2}(t'_2-t'_1)$	32.6	1.513
$15\cos\delta$		0.784
Corr. in α	$-0^{s}.54$	$n\,9.733$
Corr. in δ	$-0''.97$	$n\,9.987$

SUMMARY.

	$\alpha'-\alpha$.	$\delta'-\delta$.
	m. s.	′ ″
Uncorrected	+2 23.38	−8 58.71
Refraction	− 0.08	− 1.04
Curvature	—	− 0.09
Motion	− 0.54	− 0.97
Corrected	+2 22.76	−9 0.81

Example II. — Barnard's comet, 1885 II., observed at Harvard College observatory ($\phi = 42^\circ\ 22'.8$), July 12, 1885. Square adjusted to apparent motion as by case III. of article 10. Diagonal, $884''.0$. $\delta_0 = -7^\circ\ 42'.1$.

Observed Transits. (See article 12.)

Comparison.	Comet North.		Star North.			
	t'_1.	t'_2.	t_0.	t_1.	t_2.	t_3.
	h. m. s.	m. s.	m. s.	m. s.	m. s.	m. s.
1	16 47 28.0	47 38.6	45 42.5	45 51.2	46 42.0	46 50.5
2	50 57.5	51 8.0	49 12.4	49 21.2	50 12.0	50 20.6
3	55 25.0	55 35.3	53 39.2	53 48.7	54 38.7	54 48.0
4	59 23.4	59 35.0	57 38.5	57 48.5	58 37.9	58 47.5
5	17 4 30.2	4 46.3	2 50.2	2 56.1	3 49.5	3 55.5
6	7 59.6	8 15.5	6 20.0	6 26.2	7 20.6	7 25.5

Reduction.

Comparison.	$\frac{1}{2}(t'_2+t'_1)$.	$\frac{1}{2}(t_2+t_1)$.	$\alpha'-\alpha$.	$t'_2-t'_1$.	t_2-t_1.	Diff.
	h. m. s.	m. s.	m. s.	s.	s.	s.
1	16 47 33.30	46 16.60	+1 16.70	10.6	50.8	−40.2
2	51 2.75	49 46.60	16.15	10.5	50.8	40.3
3	55 30.15	54 13.70	16.45	10.3	50.0	39.7
4	59 29.20	58 13.20	16.00	11.6	49.4	37.8
5	17 4 38.25	3 22.80	15.45	16.1	53.4	37.3
6	8 7.55	6 53.40	14.15	15.9	54.4	38.5
Mean	16 57 43.54	56 27.72	+1 15.82	12.50	51.47	−38.97
				log		n 1.59073
Chron. corr.	−28.0	Corr. for p'	−0.12	$\log\left(-\frac{15}{2}\right)$		n 0.87506
				$\cos\delta_0$		9.99606
Sid. time	16 57 15.5	(See p. 177.)				
Red.	7 24 35					2.46185
Cam. M. T.	= 9 32 40	$\alpha'-\alpha=$	+1 15.70	$\delta'-\delta=$		+4′ 49.63

The above is a convenient form of reduction. Its three lower sections contain the determination of the local mean time of observation; the difference of right ascension, equation (1), corrected for position zero, determined on p. 177; and the difference of declination determined by (5), the declination being small.

CORRECTION FOR POSITION ZERO, EQUATIONS (32) AND (26).

$\frac{t_3+t_0}{2}$.	$\frac{t_3+t_0}{2}-\frac{t_2+t_1}{2}$.
m. s.	s.
46 16.50	−0.10
49 46.50	−0.10
54 13.60	−0.10
58 13.00	−0.20
3 22.85	+0.05
6 52.75	−0.65
	−0.18

Equation (32)	log $(-0^{s}.18)$	n 9.2553
	15 cos δ	1.1721
		n 0.4274
	g	2.9465
	p' sin $1''$	n 7.4809
Equation (26)	log (38.97)	1.5907
		n 9.0716
	Corr. of $(\alpha'-\alpha)$	$-0^{s}.12$

applied on p. 176.

From Chauvenet's Astronomy, II., equation (348), $\zeta = 50^\circ\ 12'$, $q = -\ 3^\circ\ 32'$. Then

log κ	6.443	log sin q	n 8.790
log tan ζ	0.079	log cos q	9.999
log tan$^2 \zeta$	0.158	log sin $2q$	n 9.090
log cos δ	9.996	log cos $2q$	9.997
log tan δ	n 9.131		

REFRACTION CORRECTION.

IN RIGHT ASCENSION, EQUATION (23).

tan$^2\zeta$ sin $2q$		n 9.248
$\kappa\,(d'-d)$		8.905
a. c. (15 cos δ)		8.828
$\Delta\,(\alpha'-\alpha)$	$-0^{s}.001$	n 6.981

IN DECLINATION, EQUATION (20).

tan$^2\zeta$ cos $2q$	+1.43	0.155
$\kappa\,(d'-d)$		8.905
$\Delta\,(\delta'-\delta)$	$+0''.11$	9.060

It can be seen by inspection that, on account of small declination and moderate motion, the corrections for curvature and proper motions must be insignificant; also, that the refraction correction must be very small; the latter is computed, however, to illustrate the special formula (23) for the case of adjustment to apparent motion.

VALUE OF DIAGONAL, EQUATIONS (33) AND (35).

t_3-t_0.	t_2-t_1.	Sum.
s.	s.	s.
68.0	50.8	118.8
68.2	50.8	119.0
68.8	50.0	118.8
69.0	49.4	118.4
65.5	53.4	118.9
65.5	54.4	119.9
67.50	51.47	118.97

Equation (33)	log (118.97)		2.07544
	$\frac{15}{2}$ cos δ		0.87112
		$884''.22$	2.94656
	$-g\,p^2$ sin $1''$	−0.01	
Equation (35)	$-g\,\kappa$	−0.24	
	g	= 883.97	